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Translation of "Nelineynyye zadachi kontsentratsii napryazheniy okolo otverstiy v plastinakh" OKTYABR
Teoriya Obolochek i Plastin, Trudy IV Vsesoyuznoy Konferentsii po Teorii Obolochek i Plastin, pp. 116-140, Yerevan, 24-31 ~~October~~, 1962.
Izdatelstvo ~~Akademii Nauk Armyanskoy SSR, Yerevan~~, 1964.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON SEPTEMBER 1965

NASA TT F-9549

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ABSTRACT

The author surveys 36 (mostly Soviet) papers and monographs dealing with the nonlinear problems of stress concentration near holes in plates, giving special emphasis to the solution of these problems by the methods developed by N. I. Muskhelishvili and his school for dealing with the plane problem of elasticity theory. These methods consist in the use of complex potentials in conjunction with conformal mapping and Cauchy integrals.

Savin begins with a historical and substantive survey of recent developments in the methodology of plane nonlinear elastic theory. These include the establishment of the fundamental system of equations of nonlinear elasticity theory for incompressible and compressible materials in a plane stress condition and in plane deformation and the use of the small-parameter method for the approximate solution of the system (Adkins, Green, Shield, Nicholas); the derivation of a compatibility condition for finite plane deformations of an incompressible material expressed in terms of an invariant strain characteristic (Tolokonnikov); the development of a solving system of equations for plane deformation in terms of displacements (Slezinger and Barskaya); the approximate solution of the problem of stress concentration near an elliptical hole both in the case of plane deformation and for a generalized stress condition, i.e., a thin plate in a homogeneous stress condition at infinity (Koyfman).

Following this, applications of the above techniques are discussed for a great variety of stress concentration situations: (uniaxial tension-compression, tension-compression from all sides, pure shear); stress concentration near an elliptical hole (tension along the major hole axis, tension along the minor hole axis, tension (compression) from all sides); effects of hole reinforcement by means of elastic rings; stress concentration near free and reinforced curvilinear holes.

The author concludes with a brief description of the so far unique results obtained by Ya. F. Kayuk on the pattern of the stress condition near the hole in an elastic flexible plate following its post-critical deformation.

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*/The numbers in the margin indicate pagination of the original foreign text.

§1. The development of new synthetic materials and the increased use of thin-walled constructions consisting of thin plates and shells in recent years have occasioned the need for a more exact formulation of the pertinent problems in elasticity theory. /116

This more exact formulation usually leads to the general (both physically and geometrically) nonlinear problem of elasticity theory, or to the purely geometrically nonlinear problem, or, alternatively, to the purely physically nonlinear problem.

It must be noted that the nonlinear formulation of many problems is nothing new in elasticity theory. The groundwork for nonlinear elasticity theory was laid in the very first stages of development of this science. As time went on, however, it was linear elasticity theory [32] which received the greater emphasis, yielding much that was useful from both the theoretical and the engineering standpoint.

Specifically, it was within the context of linear elasticity theory that powerful and efficient techniques were developed for solving entire classes of problems. The most effective of these were the solutions of the plane problem of elasticity theory developed by Academician N. I. Muskhelishvili [1] and his pupils, i.e., the methods of complex potentials combined with conformal mapping and Cauchy integrals, methods which afford a means of reducing the basic problems of plane elasticity theory to boundary-value problems of the theory of analytic functions of a complex variable.

These techniques also turned out to be the most effective ones in handling the nonlinear plane problem of elasticity theory -- specifically, the problem of stress concentration near holes. However, due to the enormous mathematical difficulties involved, the number of papers dealing with this problem is still very small. The list of original studies appended to the present paper contains just 20 titles.

§2. The earliest studies on the nonlinear plane problem of elasticity theory, and specifically on the concentration of stresses near a circular hole were those of Adkins, Green, and Shield [2], Adkins and Green [4], and Adkins, Green, and Nicholas [3], wherein the authors established the fundamental system of /117 equations of nonlinear elasticity theory for both compressible and incompressible materials, and for both a plane stress condition and plane deformation under the most general elasticity relations

$$\tau^{ij} = \frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_1} g^{ij} + \frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_2} B^{ij} + 2\sqrt{I_2}\, \frac{\partial W}{\partial I_3} G^{ij}, \tag{1}$$

where τ^{ij} are the contravariant components of the stress tensor referred to a curvilinear coordinate system in the deformed body; $W(I_1, I_2, I_3)$ is the deformation energy density; I_r $(r = 1, 2, 3)$ are the invariants of the deformation tensor,

$$I_1 = g^{ij} G_{ij}; \quad I_2 = I_3 g_{ij} G^{ij}; \quad I_3 = \frac{G}{g}; \quad B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs};$$
$$|G_{ij}| = G; \quad |g_{ij}| = g;$$

g^{ik} and G^{ik} are the contravariant components of the metric tensors of the undeformed and deformed states of the elastic body, respectively.

By introducing the stress function U satisfying the equilibrium equations, the authors of [2], [3], and [4] obtained the complete fundamental system of equations of the plane nonlinear problem of elasticity theory consisting of two equations for determining the two functions U, D in the case of plane deformation, and the three functions U, D, and $\lambda = h/h_0$ in the case of the generalized plane stress condition, i.e., of a thin plate, where h_0 and h are the half-thicknesses of the plate before and after deformation, respectively.

The fundamental system of equations for the generalized plane stress condition, i.e., for a thin plate, expressed in the coordinates $(z, \bar{z})$ after deformation is of the form

$$
\begin{aligned}
&\frac{\partial^2 U}{\partial z^2} - 4h_0 \frac{\sqrt{I_3}}{\lambda}\left[\frac{\partial W}{\partial J_1} + (\lambda^2-2)\frac{\partial W}{\partial J_2} - (\lambda^2-1)\frac{\partial W}{\partial J_3}\right]\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial \bar{z}} [illegible] - 1\right) = 0;\\
&\frac{\partial^2 U}{\partial z \partial \bar{z}} - 2h_0\left\{\frac{\partial W}{\partial J_1} + \left(\frac{\sqrt{I_3}}{\lambda} + \lambda^2 - 2\right)\frac{\partial W}{\partial J_2} [illegible]\right.\\
&\left. + (\lambda^2-1)\left(\frac{\sqrt{I_3}}{\lambda} - 1\right)\frac{\partial W}{\partial J_3}\right\} - 4h_0 \frac{I_3}{\lambda^2} [illegible] \left\{\frac{\partial W}{\partial J_1} \right.\\
&\left. + (\lambda^2-2)\frac{\partial W}{\partial J_2} - (\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial D}{\partial \bar{z}} \cdot \frac{\partial \bar{D}}{\partial z} = 0;\\
&\frac{\partial W}{\partial J_1} + 2\left(\frac{\sqrt{I_3}}{\lambda} - 1\right)\frac{\partial W}{\partial J_2} + \left(\frac{\sqrt{I_3}}{\lambda} - 1\right)^2\frac{\partial W}{\partial J_3} +\\
&+ 4\frac{I_3}{\lambda^2}\left(\frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3}\right)\frac{\partial D}{\partial \bar{z}} \cdot \frac{\partial \bar{D}}{\partial z} = 0;
\end{aligned}
\tag{2}
$$

where /118

$$I_3 = \frac{\lambda^2}{4a}; \quad a_{\alpha\beta} = g_{\alpha\beta}; \quad g_{33} = \frac{1}{\lambda^2};$$

$$g = \frac{a}{\lambda^2}; \quad a = |a_{\alpha\beta}|.$$

$$J_1 = \lambda^2 + 2\frac{\sqrt{I_3}}{\lambda} + 4\frac{I_3}{\lambda^2} \cdot \frac{\partial D}{\partial \bar{z}} \cdot \frac{\partial \bar{D}}{\partial z} - 3;$$

$$J_2 = 3 - 2\lambda^2 + \frac{I_3}{\lambda^2} + 2(\lambda^2 - 2)\left[\frac{\sqrt{I_3}}{\lambda} + \frac{I_3}{\lambda^2} \cdot \frac{\partial D}{\partial \bar{z}} \cdot \frac{\partial \bar{D}}{\partial z}\right];$$

$$J_3 = (\lambda^2 - 1)\left[\left(1 - \frac{\sqrt{I_3}}{\lambda}\right)^2 - 4\frac{I_3}{\lambda^2} \cdot \frac{\partial D}{\partial \bar{z}} \cdot \frac{\partial \bar{D}}{\partial z}\right].$$

The system for plane deformation in the coordinates $(z; \bar{z})$ is of the same form.

These fundamental systems can also be written in the coordinates $(\eta, \bar{\eta})$

before deformation.

It is clear from system (2) that because of the immense complexity of these equations the best one can hope for is their approximate solution.

The authors of [2], [3], and [4] suggest that such an approximate solution may be obtained by the small-parameter method whereby the functions sought, U, D, and λ, as well as all the known functions appearing in these equations are represented in the form of expansions

$$\left.\begin{aligned} U &= H_0 \varepsilon \left[U^{(1)} + \varepsilon U^{(2)} + \varepsilon^2 U^{(3)} + \cdots\right] \\ D &= \varepsilon D^{(1)} + \varepsilon^2 D^{(2)} + \varepsilon^3 D^{(3)} \\ \lambda &= 1 + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \cdots \end{aligned}\right\} \qquad (3)$$

in powers of the small parameter ε, where $D = z - \eta = u+iv$ is a complex displacement function; H_0 is a constant equal to μ, i.e., the modulus of rigidity of the material for vanishingly small deformations in the case of plane deformation and to $2h\mu$ in the case of the generalized plane state, i.e., for a thin plate.

Introducing these expansions (3) and others similar to them into the fundamental systems of equations and setting the coefficients of like powers of ε equal to zero, we obtain an infinite system of equations for the required quantities $U^{(j)}$; $D^{(j)}$; $\lambda^{(j)}$ (j = 1, 2, 3), which is the j-th approximation of the required solution of the problem.

The first approximation, i.e., the equation for the functions $U^{(1)}$, $D^{(1)}$, and $\lambda = 1$ corresponds to classical linear theory and leads to the familiar Kolosov-Muskhelishvili relations for first-order complex potentials. The method of finding these potentials is known [1]; in addition, the potentials have already been determined for many hole shapes and loading situations [9].

The fundamental systems of equations can be represented in complex coordinates both before deformation (z, $\bar{z}$) and after deformation (η, $\bar{\eta}$), so that /119 the potentials can be expressed either in the coordinates $(z, \overline{z})$ or in the coordinates $(\eta, \overline{\eta})$.

The formulas for the functions $\frac{\partial^2 U^{(2)}}{\partial \overline{z}^2}$ and $D^{(2)}$ expressed in terms of second-order potentials in the coordinates $(z, \overline{z})$ are of the form [4], [16]

$$
\begin{gathered}
\frac{\partial U^{(2)}}{\partial \overline{z}} = \varphi^{(2)}(z) + z\,\overline{\varphi'^{(2)}(z)} + \overline{\psi^{(2)}(z)} - f\,(z, \overline{z}); \\
D^{(2)} = k\,\varphi^{(2)}(z) - z\,\overline{\varphi^{(2)}(z)} - \overline{\psi'^{(2)}(z)} - f_1\,(z, \overline{z});
\end{gathered}
\tag{4}
$$

where $f(z, \overline{z})$ and $f_1(z, \overline{z})$ are known functions expressed in terms of the complex potentials $\psi^{(1)\prime}(z)$ and $\phi^{(1)\prime}(\overline{z})$ of the first approximation, i.e., in terms of known functions; k is a known constant.

The components of the stress tensor referred to the coordinates of the points of the body in the deformed state $(z, \overline{z})$ are determined from the formulas

$$
\begin{gathered}
\tau^{11} = \overline{\tau^{22}} = \sigma_{y2} - \sigma_{y1} + i2\tau_{y1y2} = -4\,\frac{\partial^2 U}{\partial \overline{z}^2}; \\
\tau^{12} = \sigma_{y1} + \sigma_{y2} = 4\,\frac{\partial^2 U}{\partial z\,\partial \overline{z}}; \\
z = y_1 + iy_2; \quad \overline{z} = y_1 - iy_2.
\end{gathered}
\tag{5}
$$

The second-order complex potentials $\phi^{(2)\prime}(z)$ and $\psi^{(2)\prime}(z)$ for the case of a finite or an infinite multiply connected region and the formulation of the fundamental boundary-value problems of the nonlinear plane problem of elasticity theory are the subjects of a paper by G. N. Savin and Yu. I. Koyfman [16].

In contrast to classical linear elasticity theory, nonlinear elasticity theory admits of fundamentally new variants of boundary-value problem formulation in the determination of second-order potentials.

Thus, the following three variants are possible for the first fundamental problem:

1) the external forces are given along a known contour of the *deformed* body;

2) the external forces are given along a known contour of the *undeformed* body;

3) the boundary of the region is given for an *undeformed* body, and the *external* forces are given along an (unknown) contour of the *deformed* body.

A similar situation is obtained in the case of the second fundemental problem [16]. Provided that the stresses are finite at infinity, the second-order /120 complex potentials $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ for the case of an *unbounded* region are of the form

where N_1 and N_2 are the values of the principal stresses at infinity; θ is the angle between the principal direction of N_1 and the Oy_1 axis; E_1 and E_2 are known constants, developed expressions for which are given in [16].

It should be noted that in one of the aforementioned papers the problem of convergence of expansions (3) for the functions U, D, and λ *is not even posed*.

No limitations are imposed on the function W, the deformation energy density, in the derivation of the fundamental system of equations of plane nonlinear elasticity theory.

In the case of incompressible materials (including rubber, which satisfies this incompressibility condition over a very wide range of elongations reaching as high as 50%), the function W in papers [2], [3], [4], and [16] is taken in the form

$$W = C_1 (I_1 - 3) + C_2 (I_2 - 3); \tag{7}$$

where $C_1 > 0$ and $C_2 > 0$ are the elastic constants of the material and are determined experimentally. The function W(z) was first proposed by Mooney [8].

In the derivation of the general equations of nonlinear plane elasticity theory and the construction of an approximate method for their solution, the authors of [2], [3], [4], and [16] impose no limitations on the function W, although it is a well-known fact that in linear classical theory the function W must obey certain very specific conditions, namely: in order for the ordinary boundary-value problems of elasticity theory to have a unique solution (Kirchhoff's theorem), the function W must be a positive definite, homogeneous quadratic form.

In nonlinear elasticity theory, on the other hand, the problem of the form of classes of functions which can serve as the function W, the deformation energy density of an elastic material, still remains unsolved [19].

The chosen function W, as we see from formula (1), has its own corresponding values of τ^{ij}, i.e., its own special elastic body.

In the general case, i.e., in the case of nonlinear elasticity theory, the function $W(I_1, I_2, I_3)$ cannot be completely arbitrary. This may be seen, for example, from the fact that for vanishingly small deformations, when all bodies obey Hooke's law, this function W must be a positive definite quadratic form.

The papers of T. Doyle and J. Ericksen [19] and Truesdell [22] contain /121 surveys of studies on this problem; Doyle and Ericksen [19] formulate the necessary conditions for an incompressible material,

$$\begin{gathered} dW > 0; \\ \frac{dW}{dI_1} + (1 + \varepsilon_i)^2 \frac{\partial W}{\partial I_2} > 0, \\ (i = 1,\ 2) \end{gathered} \tag{8}$$

which must be satisfied during the application of stresses to such a body, where δ_i are the principal deformations.

In linear theory the second condition in (8) indicates that the modulus of rigidity μ must be a positive quantity.

In 1953 Adkins and Green [2] published for the first time for a nonlinear problem an approximate (accurate to within the second approximation) solution of the Kirsch problem, i.e., the problem of the distribution of stresses near a round hole for the case of plane deformation of an incompressible material with the uniaxial stress condition $\sigma^{(\infty)} = p = \text{const}$ at infinity. Somewhat later, in paper [4], these authors, in considering other problems for a round hole, also consider the problem of the concentration of stresses near a round hole in a thin elastic (both geometrically and physically nonlinear) plate under uniaxial tension due to the forces $p = \text{const}$ at infinity. A somewhat different approach than that employed in [2-4] is used in dealing with the problem of stress concentration near a round hole in the papers of L. A. Tolokonnikov [13], I. N. Slezinger and S. D. Barskaya [14].

L. A. Tolokonnikov [13] obtained the compatibility condition for finite plane deformations of an incompressible material expressed in terms of an invariant characteristic of deformation, the deformation intensity Э$_i$.

Tolokonnikov likewise assumes that the physical law of deformation is characterized by the relationship between the octahedral tangential stress τ_i and the deformation intensity Э$_i$.

Under this assumption, proceeding on the basis of the compatibility equations and equilibrium equations which are satisfied by the introduction of the stress function, Tolokonnikov obtains the solving equation for the problem of finite plane deformations of an incompressible material in terms of the stresses.

$$k^{**} = 3\left(1 + 0.2\,\frac{p}{\mu} - \frac{p^2}{\mu^2}\right).$$

This equation is likewise integrated by the small-parameter method. He then considers (in complex coordinates of the undeformed state of the body) the problem of stress concentration near a round hole (plane deformation) whose contour was initially round, under uniaxial compression (tension); it is assumed that the physical deformation law is of the form

$$\tau_i = \mu\, tg\,\ni; \quad \ni = 2\sqrt{1.5}\,\ni_i. \tag{9}$$

In this variant, the following formula (with regard for three approximations)/122 is given for the concentration coefficient k**:

(10)

Formula (10) implies that the concentration coefficient at first increases with increasing stress for $0 < \frac{p}{\mu} < 0.2$ $(p > 0)$, and then diminishes starting with the value $\frac{p}{\mu} = 0.2$.

This is due to the increasing role played by the factor of physical nonlinearity of the material in the deformation process. The view is also expressed that consideration of two approximations basically characterizes the geometric nonlinearity of the problem.

I. N. Slezinger and S. D. Barskaya [14] obtain a solving system of equations for plane deformation in terms of displacements. It is assumed that the problem is geometrically linear and physically nonlinear, since the deformation law used is the physical one suggested by N. V. Zvolinskiy and P. M. Riz [34], [35]. In other words, it is assumed that there is a linear relationship between the principal stresses referred to the initial sectional area, and the principal elongations.

On the basis of the resulting solving equation, Slezinger and Barskaya employ the small-parameter method to obtain an approximate solution of the problem of stress concentration near a hole round (in the undeformed state) for plane deformation with a uniaxial stress condition at infinity.

The authors give formulas (up to an including second-order terms) for the stress components σ_θ and the displacement, as well as a formula and Table 1 for the values of the concentration coefficient under tension,

$$k = \frac{3+6\,(2-\chi)\,\alpha}{1+0{,}5\,(3-\chi)\,\alpha}, \qquad \alpha = \frac{p}{4\mu}; \quad \chi = \frac{\lambda+3\mu}{\lambda+\mu}, \tag{11}$$

where p is the stress at infinity referred to the initial sectional area; λ, μ are the Lamé constants.

As we see from Table 1, with this variant of the elasticity relations there is an increase in the concentration coefficient in comparison with linear theory $k^o = 3$.

The fundamental systems of equations of the nonlinear plane problem of elasticity theory obtained in [2], [3], and [4] and the approximate methods for their solution in the simplest cases of a round hole or round washer given there show that these problems can be solved for other, non-round holes or bodies only by resorting to the arsenal of powerful methods of solution developed in plane linear elasticity theory, and first of all by using the Kolosov-Muskhelishvili complex potentials in conjunction with conformal mappings and Cauchy integrals. /123

Using this method of attack, Yu. I. Koyfman [15] obtained an approximate (to within second-order terms) solution of the problem of stress concentration

Table 1

ν^* \ $\frac{p}{\mu}$	0.04	0.08	0.12	0.16	1.20
$\frac{1}{4}$	2.995	2.980	2.955	2.941	2.927
$\frac{1}{2}$	3.030	3.060	3.081	3.115	3.143

near an elliptical (and as a special case, circular) hole both in the case of plane deformation, and for the generalized stress condition, i.e., a thin plate, with a homogeneous stress condition at infinity.

In [15] Koyfman determines the first- and second-order complex potentials for various conditions at infinity and cites formulas for the stresses σ_θ^* and σ_θ^{**} on the contours of both a round and an elliptical hole; the shapes of the holes are given in the deformed and undeformed states of the body.

Some results on stress concentration near a round and an elliptical hole computed by Yu. I. Koyfman both for the case of plane deformation and for the generalized plane stress state, i.e., a thin plate, are given below.

Section 1. Stress Concentration Near a Round Hole

a) Uniaxial tension-compression.

Variant 1, where we consider the concentration of stresses over the contour of a hole which is round in the deformed state.

The stresses σ_θ^* along the hole contour in the case of tension-compression are

$$\sigma_\vartheta^* = N\left\{ 1 - 2\cos 2\vartheta + \frac{\gamma N}{4H_0}\left[(3 - k\delta) - 4\cos 2\vartheta + 4\cos 4\vartheta\right]\right\}. \qquad (12)$$

*ν is the Poisson coefficient.

where θ is the polar angle in the deformed body; N is the principal stress at infinity; γ, k, δ are elastic constants whose values are given in [4] and [16]; H_0 is equal to μ, the modulus of rigidity, for plane deformation and to $2h\mu$ for a thin plate; 2h is the thickness of the plate after deformation. The concentration coefficient is /124

$$K^* = \frac{(\overset{\bullet}{\sigma}_\vartheta)_{\max}}{N} = 3\left[1 + \frac{\gamma(11 - k\delta)}{12}\cdot\frac{N}{H_0}\right]. \tag{13}$$

In particular, for an incompressible material, by [12] and [13], we have:

for plane deformation,[1]

$$\overset{\bullet}{\sigma}_\vartheta = N\left[1 - \cos 2\vartheta + \frac{N}{4\mu}\,(1 - 2\cos 2\vartheta + 2\cos 4\vartheta)\right]; \tag{14}$$

$$K^*_{p.d.} = 3\left(1 + \frac{5}{12}\,\frac{N}{\mu}\right); \tag{15}$$

for a plate, i.e., a plane generalized stress condition,

$$\overset{\bullet}{\sigma}_\vartheta = N\left\{1 - 2\cos 2\vartheta + \frac{5+l}{32(1+l)}\cdot\frac{N}{H_0}\left[\frac{2\,(21+l)}{3\,(5+l)} - 4\cos 2\vartheta + 4\cos 4\vartheta\right]\right\}; \tag{16}$$

$$K^*_{pl.} = 3\left[1 + \frac{81 + 13l}{144\,(1+l)}\,\frac{N}{H_0}\right]; \tag{17}$$

where $l = C_2/C_1$; C_1, C_2 are Mooney constants.

Table 2 gives concentration coefficients computed using formulas (15) and (17) for various values of N/H_0. In computations with the latter formula, the value of $l = C_2/C_1$ was assumed equal to $l = 1/19$.

From the formulas for K* (15) and (17) and the data of Table 2 we see that consideration of nonlinear corrections results in a considerable divergence of

[1] Formula (14) appeared for the first time in [2].

the concentration coefficients K* from their values as given by linear theory.

Table 2

K \ N/H_0	-0.3	-0.2	-0.1	0.1	0.2	0.3	Linear theory
$K^*_{p.d.}$	2.625	2.750	2.875	3.125	3.250	3.375	3
$K^*_{pl.}$	2.514	2.676	2.838	3.162	3.344	3.486	3

In making comparison with linear theory one must bear in mind that in the present variant, the finite round contour can be obtained in two ways: either by stretching a plane with an oval hole along its minor axis, or by compress- /125 ing it along its major axis. Clearly, the larger the tension required to alter (deform) the initial oval contour into a round contour (for an oval with a shorter minor axis), the higher the stress concentration along the contour. In the case of compression, as the compressing forces are increased (for an oval elongated along the major axis), the stress concentration along the hole contour diminishes.

Variant 2, where we consider the concentration of stresses over the contour of a hole which was round prior to deformation. As à result of deformation, this round hole becomes oval. Its exact shape depends on conditions at infinity (it is elongated along the axis of tension if a tensile force is applied and along the axis perpendicular to the axis of tension upon application of a compressing force).

The stresses σ_θ^{**} along the contour of the deformed hole which was originally round are

$$\sigma_\vartheta^{**} = N\left\{1 - \cos 2\vartheta + \frac{2\gamma - k - 1}{8} \cdot \frac{N}{H_0}\left[2\gamma\frac{(3 - k\delta)}{2\gamma - k - 1} - 4\cos 2\vartheta \cos 4\vartheta\right]\right\}. \tag{18}$$

where θ is the polar angle in the undeformed plane. The rest of the symbols have the same meaning as in formula (12).

For the concentration coefficient K** formula (18) gives us

$$K^{**} = \frac{(\sigma_\vartheta)_{\max}}{N} = 3\left[1 + \frac{\gamma(11 - k\delta) - 4(k+1)}{12} \cdot \frac{N}{H_0}\right]. \tag{19}$$

For an incompressible material (18) and (19) yield for the case of plane deformation,

$$\sigma_\vartheta^{**} = N\left[1 - 2\cos 2\vartheta + \frac{N}{4\mu}(1 + 2\cos 2\vartheta - 2\cos 4\vartheta)\right]; \tag{20}$$

$$K_{\text{p.d.}}^{**} = 3\left[1 - 0{,}25\frac{N}{\mu}\right]; \tag{21}$$

and for the case of a thin plate,

$$\sigma_\vartheta^{**} = N\left\{1 - \cos 2\vartheta + \frac{17 + 29l}{96(1+l)} \cdot \frac{N}{H_0}\left[\frac{2(21+l)}{17+29l} + 4\cos 2\vartheta - 4\cos 4\vartheta\right]\right\}; \tag{22}$$

$$K_{\text{pl.}}^{**} = 3\left[1 - \frac{47 + 115l}{144(1+l)} \cdot \frac{N}{H_0}\right]. \tag{23}$$

Table 3 gives values of the concentration coefficients computed using formulas (21) and (23) for the same values of the elastic constants as in Table 2.

Comparing the numerical values given in Tables 2 and 3 we note some divergence. This can be explained by the alteration of the shape of the hole due to deformation. Thus, as the tension forces increase in Variant 2, the initially round hole becomes more and more oblate along the axis perpendicular to the axis of tension (due to the forces N at infinity), so that the stress concentration at the point $\theta = \pi/2$ of the contour diminishes. In compression, on the other hand, the round hole flattens out along the axis of compression, and /126 the stress concentration at the point $\theta = \pi/2$ of the contour increases.

Table 3

K \ N/H_0	-0.3	-0.2	-0.1	0.1	0.2	0.3	Linear theory
$K^{**}_{p.d.}$	3.225	3.150	3.075	2.925	2.850	2.775	3
$K^{**}_{pl.}$	3.315	3.210	3.105	2.895	2.790	2.685	3

b) Tension-compression from all sides.

With tension-compression from all sides, the initial round shape is unaltered while the hole changes in size, i.e., in radius, which has no effect at all on the stress condition near the hole.

In this case $\sigma_\theta^* = \sigma_\theta^{**}$, so that the concentration coefficient is given by the formula

(24)

The values of K (24) for an imcompressible material both in plane deformation and in the case of a thin plate are given in Table 4 for various values of N/H_0.

Table 4

K \ N/H_0	-0.3	-0.2	-0.1	0.1	0.2	0.3	Linear theory
$K_{p.d.}$	1.85	1.90	1.95	2.05	2.10	2.15	2
$K_{pl.}$	1.82	1.88	1.94	2.06	2.12	2.18	2

The effect of nonlinear corrections both in tension and in compression is evident from the above data.

c) Pure shear. /127

Variant 1. If the plane under consideration in the deformed state is a region weakened by a round hole while in a state of pure shear, the stresses σ_θ along the contour are

$$\sigma_\vartheta^{\bullet} = 4N\left[\cos 2\vartheta + \frac{\gamma N}{4H_0}(2 - k\delta + 4\cos 4\vartheta)\right]. \tag{25}$$

From (25) we obtain the following formula for the concentration coefficient:

$$K^{\bullet} = 4\left[1 + \frac{\gamma(6 - k\delta)}{4} \cdot \frac{N}{H_0}\right]. \tag{26}$$

For an incompressible material, (25) and (26) give us:

in the case of plane deformation,

$$\sigma_\vartheta^{\bullet} = 4N\left[\cos 2\vartheta + \frac{N}{8\mu}(1 + 4\cos 4\vartheta)\right]; \tag{27}$$

$$K^{\bullet}_{p.d.} = 4\left[1 + 0{,}625\frac{N}{\mu}\right]; \tag{28}$$

in the case of a thin plate,

$$\sigma_\vartheta^{\bullet} = 4N\left\{\cos 2\vartheta + \frac{5 + l}{32(1 + l)} \cdot \frac{N}{H_0}\left[\frac{27 - l}{3(5 + l)} + 4\cos 4\vartheta\right]\right\}; \tag{29}$$

$$K^{\bullet}_{pl.} = 4\left[1 + \frac{87 + 11l}{96(1 + l)} \cdot \frac{N}{H_0}\right]. \tag{30}$$

Table 5 contains some values of $K^*_{pl.}$ (30) and $K^*_{p.d.}$ (28) for various values of N/H_0.

Table 5

K^* \ N/H_0	0.1	0.2	0.3	Linear theory
$K^*_{p.d.}$	4.25	4.50	4.75	4
$K^*_{pl.}$	4.35	4.69	5.04	4

The foregoing results bespeak the substantial effect of nonlinear corrections of linear theory.

Variant 2. This is the case of a pure shear applied to a plate weakened by a hole which prior to deformation had the shape of a circle of a certain radius. The stresses σ_θ^{**} along the hole contour are

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$$\sigma_\vartheta^{**} = 4N\left\{\cos 2\vartheta + \frac{2\gamma - k - 1}{8} \cdot \frac{N}{H_0}\left[\frac{2\gamma(2-k\delta)}{3\gamma - k - 1} + 4\cos 4\vartheta\right]\right\}. \tag{31}$$

The concentration coefficient is given by the formula

$$K^{**} = 4\left[1 + \frac{\gamma(6-k\delta) - 2(k+1)}{4} \cdot \frac{N}{H_0}\right] \tag{32}$$

For an incompressible material we have:

in the case of plane deformation,

$$\sigma_\vartheta^{**} = 4N\left[\cos 2\vartheta + \frac{N}{8H_0}(1 - 4\cos 4\vartheta)\right]; \tag{33}$$

$$K_{p.d.}^{**} = 4\left[1 - 0{,}375\frac{N}{\mu}\right]; \tag{34}$$

in the case of a thin plate,

$$\sigma_\vartheta^{**} = 4N\left\{\cos 2\vartheta + \frac{17+29l}{96(1+l)} \cdot \frac{N}{H_0}\left[\frac{27-l}{17-29l} - 4\cos 4\vartheta\right]\right\}; \tag{35}$$

(36)

Table 6

K^{**} \ N/H_0	0.1	0.2	0.3	Linear theory
$K_{p.d.}^{**}$	4.15	4.30	4.45	4
$K_{pl.}^{**}$	4.19	4.37	4.56	4

From the data in Tables 5 and 6 we see the effect of changes in the original shape of the hole on the stress concentration coefficients.

Analysis of the stress condition near the hole in this case reveals that the distribution of stresses over the contour of the hole is considerably different from that in Variant 1. Thus, as the load is increased, the stress concentration at the point $\theta = 0$ of the contour diminishes, and $(\sigma_\theta^{**})_{max}$ is attained at the point $\theta = \pi/2$. In contrast to Variant 1, the maximum* stresses are compression stresses.

Section 2. Stress Concentration Near an Elliptical Hole /129

a) Tension along the major axis of an elliptical hole.

Variant 1. In the deformed state, the hole is an ellipse with major semiaxis a and minor semiaxis b.

In the present section we cite data for the case of plane deformation of an incompressible material, where $m = \frac{a - b}{a + b} = \frac{1}{3}$.

The concentration coefficient $K^*_{p.d.}$ is

$$K^*_{p.d.} = 2\left(1 + 0{,}225\frac{N}{\mu}\right). \tag{37}$$

It is clear from (37) that consideration of nonlinearity leads to an increase in the concentration coefficient in comparison with linear theory $K_0 = 2$ as the tensile load increases, and to its reduction as compression forces increase.

Variant 2. This is the case of stress concentration near a hole of elliptical shape prior to deformation.

Here the concentration coefficient is given by

$$K^{**}_{p.d.} = 2\left(1 - 0{,}163\frac{N}{\mu}\right). \tag{38}$$

*In absolute value.

For the same values of N/μ, formula (38) gives smaller values of $K^{**}_{p.d.}$ than does formula (37). This smaller value of the concentration coefficient $K^{*}_{p.d.}$ (38) as compared with $K^{*}_{p.d.}$ (37) under tension is due to the fact that tension flattens the hole, and its curvature at the point $\theta = \pi/2$ diminishes. Under compression the shape of the hole more nearly approximates a circle, so that the stress concentration coefficient increases.

b) <u>Tension along the minor axis of an elliptical hole.</u>

<u>(for m = 1/3)</u>

<u>Variant 1</u>. The concentration coefficient is determined from the formula

$$K^{*}_{p.d.} = 5\left(1+0{,}463\,\frac{N}{\mu}\right). \tag{39}$$

Linear theory in this case yields a value of K = 5. As is clear from (39), for $N/\mu = 0.3$ the value of $K^{*}_{p.d.} = 5.693$, and for $N/\mu = -0.3$ we have $K^{*}_{p.d.} = 4.306$. The deviation from linear theory is evidently quite sizeable. /130

<u>Variant 2</u>. The stress concentration coefficient is

$$K^{**}_{p.d.} = 5\left(1-0{,}063\,\frac{N}{\mu}\right). \tag{40}$$

For $N/\mu = \pm 0.3$ the values of K (40) are $K^{**}_{p.d.} = 4.91$ and $K^{**}_{p.d.} = 5.09$, respectively, which diverge but very slightly from the value K = 5 given by linear theory.

c) <u>Tension (compression) from all sides of a plane with an elliptical hole.</u>

<u>Variant 1</u>. The concentration coefficient is given by the formula

$$K^{*}_{p.d.} = 4\left(1+0{,}688\,\frac{N}{\mu}\right). \tag{41}$$

<u>Variant 2</u>. The concentration coefficient may be determined from

$$K^{**}_{p.d.} = 4\left(1-0{,}063\,\frac{N}{\mu}\right). \tag{42}$$

Numerical values computed using formulas (41) and (42) for some values of N/μ are given in Table 7.

Table 7

K \ N/μ	Linear theory	-0.3	-0.2	-0.1	0.1	0.2	0.3
$K^{*}_{p.d.}$	4	3.18	3.45	3.73	4.28	4.55	4.83
$K^{**}_{pl.}$	4	4.08	4.05	4.03	3.98	3.95	3.93

As we see from Table 7, Variant 1 involves substantial deviations from the value $K = 4$; this, of course, is as expected.

The above results allow us to draw certain general conclusions.

1. In nonlinear theory, the stress concentration coefficient depends both on the type and the magnitude of the load at infinity, and on the elastic properties of the material and type of elastic equilibrium involved, i.e., plane deformation or a generalized stress condition (the case of a thin plate).

2. Increases (within a certain interval) in the tensile forces required for the deformation of the initial contour into a circle produce an increase in the stress concentration coefficient on the (deformed) contour in the case /131 of an incompressible material; increases in the compression forces result in a reduction of the coefficient.

3. The deviation of the concentration coefficient from its value given by linear theory is generally speaking quite significant. The largest difference is observed in the case of pure shear.

4. For Variant 1, i.e., when the shape of the hole contour is given for the deformed state of the body, the stress concentration coefficient diminishes with increasing uniaxial tension and increases with uniaxial compression. These

changes may be explained by distortion of the shape of the initial contour into the given shape in the course of deformation.

A particularly sharp change in the distribution of stresses over the hole contour is associated with pure shear.

Table 8 contains expressions for the concentration coefficients K for a plane weakened by an elliptical hole for all of the problems considered above.

Table 8

Type of loading	Variant	Tension-compression along the major axis	Tension-compression along the minor axis	Tension-compression from all sides
Tension	$4\left(1+0.688\frac{N}{\mu}\right)$	$2\left(1+0.255\frac{N}{\mu}\right)$	$5\left(1+0.463\frac{N}{\mu}\right)$	
	$4\left(1-0.063\frac{N}{\mu}\right)$	$\left(1-0.163\frac{N}{\mu}\right)$	$5\left(1-0.063\frac{N}{\mu}\right)$	
Compression	$4\left(1-0.688\frac{N}{\mu}\right)$	[illegible]$0.224\frac{N}{\mu}\Big)$	$5\left(1-0.463\frac{N}{\mu}\right)$	
	2 $4\left(1+0.063\frac{N}{\mu}\right)$	[illegible]$0.163\frac{N}{\mu}\Big)$	$5\left(1+0.063\frac{N}{\mu}\right)$	
Tension-compression	Linea theory	4		

Section 3. The Effect of Hole Reinforcement by Means of Elastic Rings

In their paper [16] G. N. Savin and Yu. I. Koyfman consider the problem of reinforcing the edge of a round hole by means of an elastic ring plate consisting of a different, generally also nonlinear, material pressed or welded in the hole when the plate is under a homogeneous stress condition at infinity.

As illustrations, the authors give formulas for the stresses σ_r, σ_θ on the contact or weld seam contour for an infinite plane with a round hole (for the case of plane deformation of an incompressible material) press-fitted or weld-reinforced with an absolutely rigid round ring (washer). These formulas /132 imply that absolutely rigid reinforcement reduces the concentration of stresses along the hole contour.

In another paper [17], Yu. I. Koyfman solves the problem of reinforcing a round hole in an infinite plate with the aid of a ring made of thin elastic rod stock of constant cross section welded into the hole; the elastic equilibrium of the ring is described by equations from the theory of small deformations of thin curvilinear rods.

The formulas and tables for the stress components along the contour included in [17] indicate that a reinforcing ring in the form of a linearly elastic curvilinear rod substantially reduces the concentration of stresses along the hole contour.

Section 4. Stress Concentration Near Free and Reinforced Curvilinear Holes

Summarizing studies in this most general (from the standpoint of nonlinear elasticity theory) field of research on stress concentration near holes as delineated in papers [2, 3, 4, 15, 16, 17], we see that the application of the powerful methods of plane linear elasticity theory, and, specifically, of the Kolosov-Muskhelishvili complex potentials in conjunction with conformal mapping and Cauchy integrals offers every possibility for obtaining effective approximate (up to and including the second approximation) solutions of problems on the concentration of stresses near unreinforced as well as reinforced curvilinear holes.

Reinforcement of such holes can be effected in two ways: by means of a wide elastic ring-plate, or with a narrow thin elastic ring, i.e., as is done in [16] and [17] for a round hole.

It is of course of considerable interest to the scientist and engineer to be able to reinforce holes with rings such that the stress concentration near them is either totally eliminated or at least minimized.

§3. As we know, the usual distribution of stresses is usually altered near holes due to the appearance around them of so-called stress concentration zones. The stresses in these zones can reach relatively large values. This is especially true of holes with angular points of small radii of curvature. At such angular points the stresses may come to exceed the elastic limit of the material and in the case of plastic materials may reach the yield point.

For many materials the tension-compression curve deviates from a straight line (Hooke's law) even with relatively small stresses. For such materials as the non-ferrous metals, certain plastics, et al., this curve departs from the straight curve of Hooke's law rather markedly.

For the great majority of materials, the uniaxial tension-compression curve is of the shape shown in Figure 1. /133

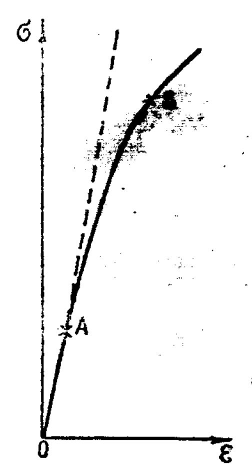

Figure 1.

Let us suppose that a plate weakened by some small curvilinear hole is in a uniaxial stress condition with stresses $\sigma = p = \text{const}$ at infinity.

At points sufficiently distant from the hole the stresses are equal to p, which corresponds to the point A (see Figure 1.) on the deformation curve; in the stress concentration zone near the hole, the stresses may lie at point B and above (Figure 1), i.e., at those points of the curve where the latter deviates slightly perhaps, but nevertheless noticeably from the straight curve of Hooke's law.

We are now faced with the following question: How can these small departures from Hooke's law affect the value of the concentration coefficient, as well as the magnitude of the stress concentration zone near a hole, and especially a hole with rounded corners of small radii of curvature?

In order to answer this question, we must compare the results of two solutions for the same case: the results of the classical solution, i.e., that based on linear elasticity theory, and the results of the solution which takes into account the above deviations from Hooke's law.

Hence we see the necessity of knowing the stress concentrations near curvilinear holes in a physically nonlinear (slightly divergent from linearity) plane field. In other words, we see the necessity of solving problems where in the fundamental equations of classical elasticity theory the linear dependence between stresses and strains -- Hooke's law -- is replaced by a nonlinear law which becomes the usual Hooke's law for small deformations (e.g., for metals under deformations not exceeding 0.1%).

On such method of solving these problems on stress concentrations near holes is examined in paper by G. N. Savin [20]. Savin adopts the simplest variant of the nonlinear dependence between stresses and deformations suggested by ...derer [21]

$$\left.\begin{aligned} \varepsilon_x &= \frac{\sigma_0}{3K} + \frac{g(t_0^2)}{2G}(\sigma_x - \sigma_0); \\ \varepsilon_y &= \frac{\sigma_0}{3K} + \frac{g(t_0^2)}{2G}(\sigma_y - \sigma_0); \\ \psi_{xy} &= \frac{g(t_0^2)}{G}\tau_{xy}; \end{aligned}\right\} \tag{43}$$

As experiments show, the dimensionless constant g_2 (44) for the non-ferrous metals (copper, copper alloys, etc.) is of the same order of magnitude as the moduli K and G expressed in $\mathrm{Kg/cm^2}$, i.e., 10^5–10^6; for this reason, $\lambda([\lambda]\ \frac{\mathrm{cm}^4}{\mathrm{kg}^2})$ /134 for these materials is of the inverse order of magnitude of the moduli K and G, i.e., 10^{-5}–10^{-6}. For pure copper, for example, tension experiments with stresses of $\sigma \leq 1000\ \mathrm{kg/cm^2}$ yield values of

$$K = 1{,}37 \cdot 10^6\ \frac{\mathrm{kg}}{\text{см}^2};\quad G = 0{,}46 \cdot 10^6\ \frac{\mathrm{kg}}{\text{см}^2};\quad g_2 = 0{,}18 \cdot 10^6 \text{ и } \lambda = 0{,}255 \cdot 10^{-6}\ \frac{\text{см}^4}{\mathrm{kg}^2}. \tag{49}$$

The presence in equation (46) of the parameter λ (47) of small absolute value naturally suggests that the solution of this equation might be sought in the form of the expansion

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(50)

where H_0 is a constant equal to

$$H_0 = \frac{1}{\beta} = \frac{G}{g_2}\sqrt{3 + \frac{G}{K}}, \quad \varepsilon = \frac{1}{g_2}.$$

Substituting this expansion of the function F(x, y) into the basic equation (46) and equating to zero the coefficients of like powers of ε, we obtain an infinite system of nonlinear differential equations (51) and (52) for determining the functions $F^{(0)}(x, y)$ and $F^{(k)}(x, y)$ $(k = 1, 2, 3, \ldots)$

(51)

(52)

where A_k is a nonlinear operator over the functions and their derivatives of the previous approximations, i.e., over the functions $F^{(0)}$; $F^{(1)}$; $F^{(2)}, \ldots, F^{(k)}$ and their derivatives. The expanded form of the operator A_1 is given in [21] and [20], and that of operator A_2 is given in a paper by Jindra [23].

As we see, the biharmonic function $F^{(0)}(x, y)$ can be found by employing the entire powerful apparatus of plane linear elasticity theory [1]. The function $F^{(0)}$ is determined from the given forces along the hole contour and the given conditions "at infinity". The functions $F^{(1)}(x, y)$, $F^{(2)}(x, y)\ldots$, on the other hand, are determined from non-homogeneous equations (52) for zero values of the forces along the hole contour and a zero stress condition at infinity.

Clearly, the principal difficulty in solving the problem at hand consists in finding some particular solution of equation (52). In obtaining the general solution of homogeneous equation (52) with the corresponding boundary conditions and conditions at infinity it is convenient to make use of the powerful apparatus of classical (linear) plane problem of elasticity theory developed by the school of Academician N. I. Muskhelishvili [1].

The problem of stress distribution near a round hole in a thin plate consisting of a physically nonlinear (by elasticity relations (43)) material is solved approximately (to within the second approximation) for the first time by Jindra in his paper [23], which contains formulas for the stress components in the region around the hole, and specifically along the hole contour.

For the concentration coefficient K in the indicated approximation we have the formula

$$K = \frac{\sigma_\theta}{p} = 3\ (1 - 3{,}45\ \lambda p^2). \tag{53}$$

The second term in (53) represents a correction for the physical nonlinearity of the material. Thus, for copper with $p \cong 333.3$ this correction turns out to be about 10%, i.e., large enough to be considered in computations.

Subsequently, I. A. Tsurpal [24-29] considered a number of new problems /136 in the same formulation. Thus, in [24] he examines the problem of the concentration of stresses near a round hole in a nonlinearly elastic plate under tension from all sides. The solution is given in the third approximation, i.e., the stress function is taken in the form

$$F(x, y) = H_0 [F^{(0)}(x, y) + \varepsilon F^{(1)}(x, y) + \varepsilon^2 F^2(x, y)].$$

For the stress concentration coefficient K, Tsurpal obtains the formula

$$K = \left(\frac{\sigma_\theta}{p}\right)_{r=R} = 2\,[1 - 1.5\,\lambda p^2 + 10.605\,\lambda^2 p^4]. \tag{54}$$

From (54) we see that in the first place the stress concentration coefficient K depends nonlinearly on the elastic properties of the plate material and on the magnitude of the external load p.

This special problem is used to investigate the rate of convergence of the method of successive approximations, i.e., to compare the values of the concentration coefficient in the first, second, and third approximations for a copper plate with elastic constants (49). The results of this comparison are given in Table 9.

Table 9

p, kg/cm^2	Concentration coefficient		
	Linear theory	Nonlinear theory	
	First approximation	Second approximation	Third approximation
100	2.000	1.990	1.990
200	2.000	1.988	1.988
300	2.000	1.985	1.985
450	2.000	1.847	1.899

We see from the table that at least for axisymmetrical stress conditions, already the second approximation yields a degree of accuracy sufficient for engineering purposes.

Paper [25] contains a study of the stress condition of a hollow cylinder under uniform external and internal pressure (plane deformation); the author takes into account the aforementioned (45) physical nonlinearity of the tube material.

In [26] Tsurpal examines a plate with a round hole under conditions of pure shear at infinity. For the concentration coefficient (in the second approximation) he obtains the formula

$$K=\left(\frac{\sigma_{\vartheta}}{\tau}\right)_{r=R}=-4\sin 2\vartheta+\lambda\tau^{2}\,(17.38\sin 2\vartheta-6.2\sin 6\vartheta). \tag{55}$$

Formula (55) implies that even with a small departure of the elastic law from Hooke's law the concentration coefficient does not remain constant, but depends quite considerably on the magnitude of the external load and on the elastic properties of the plate material.

Paper [28] is concerned with the determination of the elastic constants /137
K, G, and g_2 for certain materials.

In his paper presented at this Conference, I. A. Tsurpal examines the contact problem on the reinforcement of a round hole by means of an elastic ring of another material. In its formulation this problem is similar to that solved by Yu. I. Koyfman and G. N. Savin in [16]. In this paper [29] I. A. Tsurpal is concerned with the effect of physical nonlinearity on the stress condition in a plate when the hole is reinforced by an elastic ring of a different material or by an absolutely rigid ring, or, finally, by an elastic washer welded into the hole. All of these cases are considered both for all-sided and for uniaxial stress conditions of the plate at infinity.

The results of the studies of stress concentration near a round hole carried out in [24-27] indicate that with elasticity relations (43), i.e., with this variant of physically nonlinear theory and the chosen degree of accuracy (up to and including the second approximation) there is a reduction of the stress concentration coefficient in comparison with linear theory.

Just as in the general case of the nonlinear plane problem discussed in §2 of the present survey, there are no studies available on the convergence of the method of successive approximations for the physically nonlinear problems considered in §3. Until the present time, no paper has appeared on the stress condition near a noncircular hole under elasticity relations (43), despite the fact that it would be of great interest to investigate the effect of the physical nonlinearity of the material on the concentration of stresses at angular points of hole contours where radii of curvature are quite small.

As we know from [10], the method of solving physically nonlinear problems under elasticity relations (43) as set forth in [16] permits one to undertake the solution of some problems during the elastic-plastic stage of deformation in the stress concentration zone.

§4. If a flexible elastic plate whose material obeys Hooke's law is weakened by some small hole and is subjected to the action of a uniform compression load applied along the outer plate contour, prior to losing stability, i.e., prior to buckling, the plate will be in a plane stress condition. In this case a stress concentration zone arises near the hole, and this can be determined by the known methods.

The problem of stress concentration near a hole in a flexible elastic plate when it loses stability and enters the phase of post-critical deformations is of considerable theoretical and practical interest.

In other words, it is very desirable to know the pattern of the stress /138 condition near a hole in a flexible elastic plate in the stage of post-critical deformation.

Unfortunately, we know of no published studies on this problem. The only papers in this area are by Ya. F. Kayuk [30] and [31], who is scheduled to address the present Conference. For this reason, I shall touch but briefly on his results, leaving a more thorough discussion to the author himself.

In papers [30] and [31] the plate material is assumed to be elastic and to obey Hooke's law. The author considers the axially symmetrical post-critical deformation of a thin round plate weakened by a small round hole and acted upon by a uniform compression load applied to its outer contour under the following conditions: 1) when the external contour of the plate is hinge supported, and 2) when it is rigidly fixed. The inner contour -- the hole contour -- is free of external forces. In order to evaluate the variation of stress concentration near the hole and for purposes of comparison, the author likewise considers the problem of the post-critical deformation of a similar solid plate. The solution of the problems posed is reduced to the integration of Kármán nonlinear equations. The equations are solved with the aid of the small-parameter method.

Kayuk's results are summarized in Tables 10 and 11.

Table 10 indicates the membrane forces $T^0_{\theta\theta}$ on the free (inner) contour of the ring plate.

Table 10

$\varepsilon = \frac{\Delta P}{P_{cr.}}$	0.00	0.10	0.30	0.50	0.70
Hinge support $(T^0_{\theta\theta})\rho = 0.10$	-2	-2.13	-2.12	-1.92	-1.69
Rigid fixed support $(T^0_{\theta\theta})\rho = 0.12$	-2	-2.20	-2.41	-2.67	-2.90

Table 11 gives the values of the moments $M^0_{\theta\theta}$ on the free (inner) contour of the ring plate and along the corresponding contour of the solid plate.

Table 11

	$M^0_{\theta\theta}$	
Plate	Hinge support, $\rho = 0.10$	Rigid fixed support, $\rho = 0.12$
Solid	$5.41\eta + 6.29\eta^3$	$13.66\eta - 6.20\eta^3$
Ring	$14.1\eta + 6.11\eta^3$	$19.67\eta + 3.96\eta^3$

where $\eta = \dfrac{\Delta P}{P_{cr.} + \Delta P}$. /139

From the data of Tables 10 and 11 we see that in the case of a plate hinge-supported along its outer contour, increasing post-critical deformations are accompanied by a reduction of the concentration of membrane forces $T^0_{\theta\theta}$ on its free (inner) contour and by an increase in the concentration of the moments $M^0_{\theta\theta}$ in comparison with those on the corresponding (mentally isolated) contour $\rho = 0.10$ of a solid plate.

In the case of plate rigidly fixed along its outer contour, increasing post-critical deformations are accompanied by an increase in both the membrane forces $T^0_{\theta\theta}$ and the moments $M^0_{\theta\theta}$ on the free contour $\rho = 0.12$.

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Translated for the National Aeronautics and Space Administration by the FRANK C. FARNHAM COMPANY.